Spectral Technique in Atmospheric Modelling

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Meteorological Training Institute, India Meteorological Department (IMD), Pune

R. Krishnan

Centre for Climate Change Research

Indian Institute of Tropical Meteorology, Pune

1st lecture: Slides 01-21
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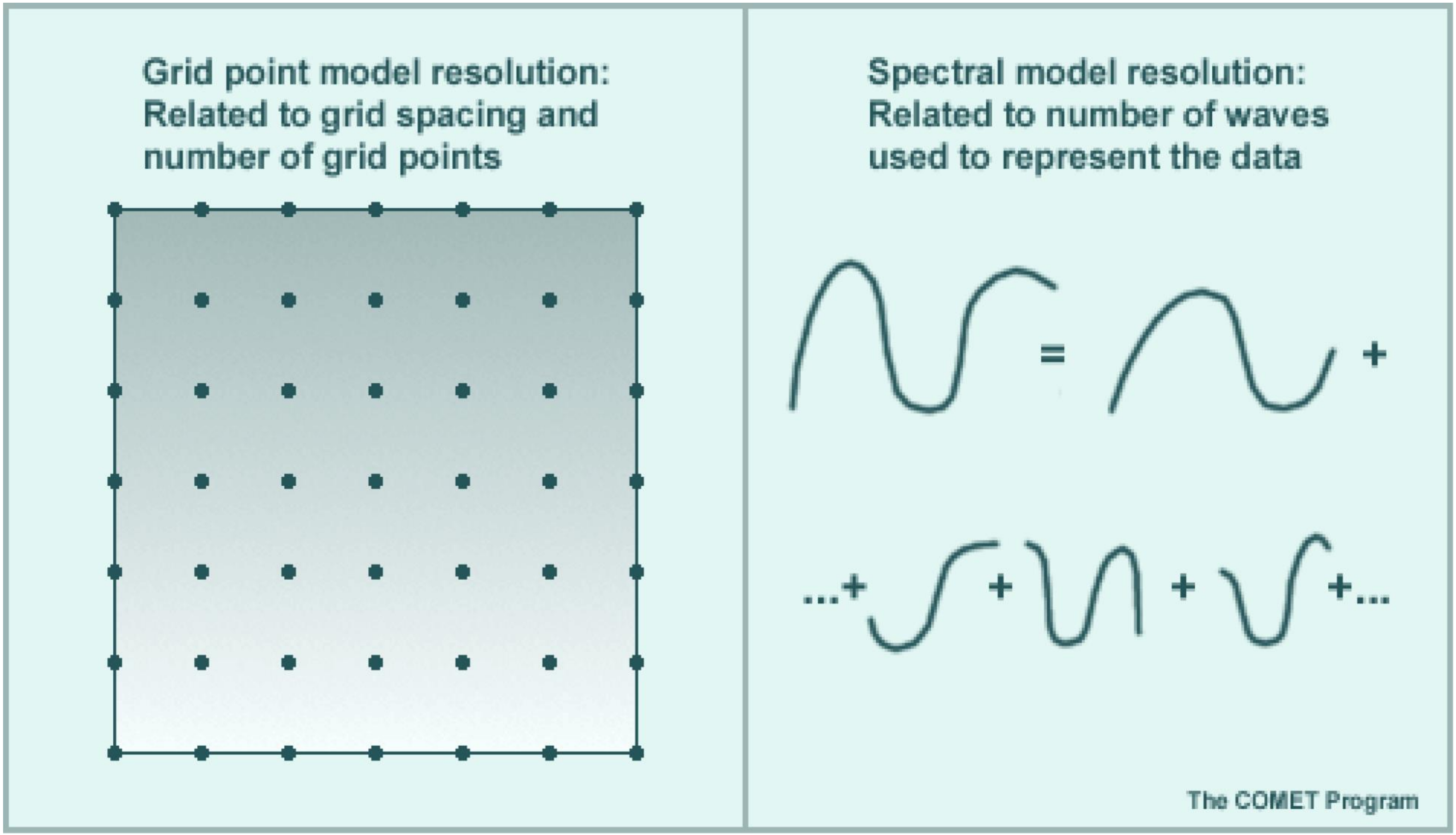
3rd lecture: Slides 41-62

Formulation of Atmospheric General Circulation Models (AGCM)

Grid-point method : The atmospheric dynamical variables in space and time are solved on a finite difference grid

Spectral method: Here, the atmospheric dynamical fields over the globe are represented in the form of waves using spherical harmonics. The spectral technique comes under the class of Galerkin methods and is very useful in numerically solving linear and nonlinear (partial) differential equations


Grid point model resolution:
Related to grid spacing and
number of grid points
Spectral model resolution:
Related to number of waves
used to represent the data
=
+
...+
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The COMET Program

An Introduction to Global Spectral Modeling

2nd Revised and Enlarged Edition

T.N. Krishnamurti
H.S. Bedi
V.M. Hardiker
L. Ramaswamy

Atmospheric and Oceanographic Sciences Library

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History

•Haurwitz (1940) initiated the use of the spherical harmonics to solve two-dimensional non-divergent vorticity equation in the spherical domain.

•Silberman (1954) extended the treatment of nonlinear advection terms using the interaction coefficient method, which requires large memory allocations and therefore it was an undesirable proposition.

•Later Platzman (1960), Baer and Platzman (1961), and Kubota et al. (1961), Ellsaesser (1966), Robert (1966) and others analysed many of the mathematical characteristics of spectral technique.

•Eliasen et al. (1970) and Orszag (1970) independently developed the transform method for evaluation of the nonlinear terms. The transform method ultimately popularized the spectral method for studies of general circulation and numerical weather prediction at the operational and research centers (Bourke, 1972, 1974, 1977; Eliasen and Machenhauer, 1974, Sela, 1980).

Advantages

- Accurate evaluation of the nonlinear advection terms
- Nice conservation properties (eg. Spectral methods of solving barotropic non-divergent models conserve area-averaged mean square kinetic energy and mean square vorticity – two quantities that fail to be conserved with some finite different methods).
- Ease of modelling flow over the entire globe. Mapping of the sphere in spectral models automatically makes a more uniform grid spacing that is common with finite difference models.
- One of the problems with finite difference approach is that near the poles, constant longitudinal increments, yield small geographical east-west distances between grid-points, and these very small distances limit the time step due to CFL criterion. The spectral technique is free from polar singularities
- Easy implementation of semi-implicit time integration schemes
- Fewer computations as compared to grid-point models at the same horizontal resolution.
- Accurate and stable numerics

Disadvantages

- Difficulties in handling discontinuity – Gibbs phenomena

4.3 A Meteorological Application

Suppose we want to represent the observed temperature field at 850 mb around a latitude circle using wave-like functions. We use a sum of sine and cosine functions to do this. This gives us the finite discrete Fourier series as

$$T_i = A_0 + \sum_{k=1}^{n} A_k \cos\frac{ik\pi}{n} + \sum_{k=1}^{n-1} B_k \sin\frac{ik\pi}{n}, \tag{4.15}$$

where n is the total number of wave components used to describe the temperature field. The points at which the temperature is known are represented by $i = 0,\ 1, 2, 3,\ \ldots\ , 2n-1$.

If T is defined at 100 points, then we will have a total of 50 waves. The integer k is called the *zonal wavenumber*, which denotes the number of waves along a latitude circle. Thus $k = 10$ implies a wavelength of $L = 360°/10 = 36°$. The coefficients A_k and B_k are obtained using the following formulas, i.e., the *discrete Fourier transform*:

$$A_k = \frac{1}{n}\sum_{i=0}^{2n-1} T_i \cos\frac{ik\pi}{n}, \qquad B_k = \frac{1}{n}\sum_{i=0}^{2n-1} T_i \sin\frac{ik\pi}{n}. \tag{4.16}$$

The temperature data are used to obtain A_k and B_k. Furthermore, (4.15) can be used along with the above equation in order to reconstruct the original temperature field if we know the coefficients.

Fourier representation and FFT

Representation of any field around latitude circles as a sum of sine and cosine functions. Computation of the Fourier Harmonics using Fast Fourier Transforms (FFT)

$$T_j(\lambda) = A_0 + \sum_{k=1}^{n} A_k \cos\frac{jk\pi}{n} + \sum_{k=1}^{n-1} B_k \sin\frac{jk\pi}{n}$$

$$A_k = \frac{1}{n}\sum_{j=1}^{2n-1} T_j \cos\frac{jk\pi}{n} \qquad B_k = \frac{1}{n}\sum_{j=1}^{2n-1} T_j \sin\frac{jk\pi}{n}$$

Inverse FFT allows reconstruction of the field around latitude circles using the Fourier Harmonics

$$f(\lambda) = \sum_{k=-\infty}^{\infty} F(k) e^{ik\lambda} \qquad 0 \le \lambda \le 2\pi$$

The functions used in most global atmospheric spectral models as the basis functions are the *spherical harmonics*, a combination of sine and cosine functions that represent the zonal structure and associated Legendre functions that represent the meridional structure. Moreover, we work with the spherical domain. This is unlike the grid-point finite-difference models, where we generally work with a limited area in the Cartesian coordinate system.

Using the above basis functions, a dependent variable, say $A(\lambda,\phi)$, can be represented as

$$A(\lambda,\phi)=\sum_{m=-j}^{j}\sum_{n=|m|}^{j+|m|}A_n^m e^{im\lambda}P_n^m(\sin\phi), \tag{4.17}$$

where m is the zonal wavenumber, n is the two-dimensional (total) wavenumber, j is the maximum wavenumber resolved, λ is the longitude, ϕ is the latitude, A_n^m is the spectral coefficient, and P_n^m is an associated Legendre function of the first kind. A more detailed

Spectral Representation

$$f(\lambda,\mu,\sigma,t)=a\sum_{m=-J}^{J}\sum_{l=|m|}^{K} f_l^{\,m}P_l^{\,m}(\mu)e^{im\lambda} \qquad l>|k|$$

where $\mu=\sin\phi$; $f_l^{\,m}$ are the spherical harmonic coefficients, also known as spectral coefficients and are complex in nature; $P_l^m(\mu)$ is the associated *Legendre* polynomial, of degree *l* and order *m*; *a* is the radius of the earth; *m* represents zonal wave number; *l* – *m* denotes meridional wave number; *J* is the wave number truncation and *K* is the highest degree of associated *Legendre* polynomial (depends on the Truncation scheme used). The factor $e^{im\lambda}$ describes the east-west variation, and the factor $\boldsymbol{P_l^{\,m}}$ describes the north-south variation of the spherical harmonic wave.

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Wave equation

The **wave equation** is a second-order linear partial differential equation for the description of waves—as they occur in classical physics—such as mechanical waves (e.g. water waves, sound waves and seismic waves) or light waves. It arises in fields like acoustics, electromagnetics, and fluid dynamics. Due to the fact that the second order wave equation describes the superposition of an incoming wave and an outgoing wave (i.e. rather a standing wave field) it is also called "Two-way wave equation" (in contrast, the 1st order One-way wave equation describes a single wave with predefined wave propagation direction and is much easier to solve due to the 1st order derivatives).

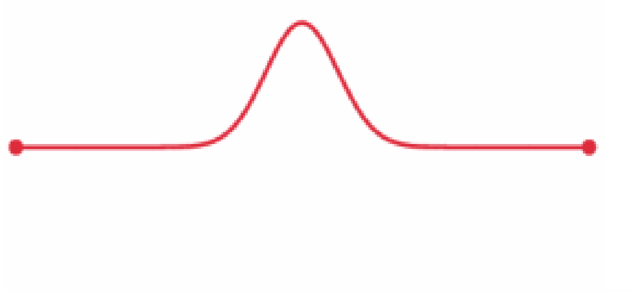
Historically, the problem of a vibrating string such as that of a musical instrument was studied by Jean le Rond d'Alembert, Leonhard Euler, Daniel Bernoulli, and Joseph-Louis Lagrange.[1][2][3][4][5] In 1746, d'Alembert discovered the one-dimensional wave equation, and within ten years Euler discovered the three-dimensional wave equation.[6]

The scalar wave equation is

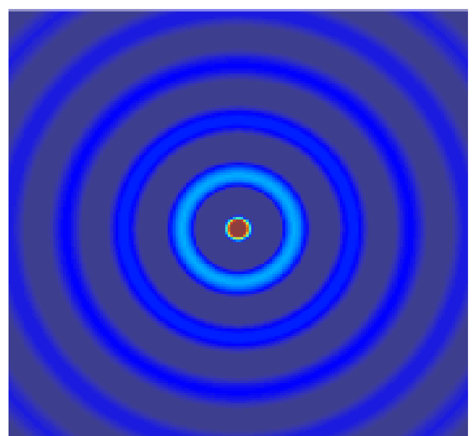
$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} \right)$$

where c is a fixed non-negative real coefficient

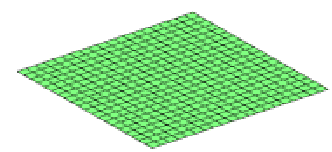
$$\ddot{u} = c^2 \nabla^2 u$$



A pulse traveling through a string with fixed endpoints as modeled by the wave equation.



Spherical waves coming from a point source.



A solution to the 2D wave equation

Chapter 6

Mathematical Aspects of Spectral Models

6.1 Introduction

In this chapter we provide an introduction to the topic of spherical harmonics as basis functions for a global spectral model. The *spherical harmonics* are made up of trigonometric functions along the zonal direction and associated Legendre functions in the meridional direction. A number of properties of these functions need to be understood for the formulation of a spectral model. This chapter describes some useful properties that will be used to illustrate the procedure for the representation of data sets over a sphere with spherical harmonics as basis functions. The calculations of Fourier

Consider the equation

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2},$$

which is satisfied by the velocity potential of a compressible fluid. Here u represents the velocity potential and c is the speed of gravity waves in the fluid. If the fluid is in a steady state, this equation reduces to *Laplace's equation*, which is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 . \tag{6.1}$$

Source: Krishnamurti et al. 2006

are of fundamental importance in the study of fluid dynamics. The solutions of Laplace's equation in a spherical coordinate system are the spherical harmonics, and are obtained by the method of *separation of variables*. That is, the solutions can be broken up into factors, each factor being the function of a single coordinate.

The transformation between Cartesian and spherical coordinates is given by

$$x = r\cos\lambda\,\sin\theta, \qquad y = r\sin\lambda\,\cos\theta, \qquad z = r\cos\theta\,, \tag{6.2}$$

where r is the radius, λ is longitude, and θ is co-latitude. Substituting x, y, and z from (6.2) into (6.1), we obtain Laplace's equation on a sphere:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \lambda^2} = 0\,, \tag{6.3}$$

or

$$\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\lambda^2}\right] = 0\,.$$

In the case of atmospheric models, latitude is usually used instead of co-latitude as one of the coordinates. Hereafter, θ denotes the latitude. Laplace's equation (6.3) in the (r, λ, θ) coordinate system (where θ is latitude and remembering sin(colat) = cos(lat)) then takes the form

$$\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\cos^2\theta}\frac{\partial^2 u}{\partial\lambda^2}\right] = 0\,,$$

or

$$\left.\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\cos^2\theta}\frac{\partial^2 u}{\partial\lambda^2}\right] = 0\,. \tag{6.4}$$

We use the method of separation of variables to solve (6.4). For this, assume a solution of the form

$$u = R(r)L(\lambda)P(\theta) \tag{6.5}$$

where R is a function of r, L is a function of λ, and P is a function of θ. Substituting (6.5) into (6.4) we obtain

$$LP\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{RL}{\cos\theta}\frac{d}{d\theta}\left(\cos\theta\frac{dP}{d\theta}\right)+\frac{RP}{\cos^2\theta}\frac{d^2L}{d\lambda^2}=0,$$

or, on dividing by RLP, we get

$$\frac{1}{P\cos\theta}\frac{d}{d\theta}\left(\cos\theta\frac{dP}{d\theta}\right)+\frac{1}{L\cos^2\theta}\frac{d^2L}{d\lambda^2}=-\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right). \tag{6.6}$$

The left-hand side of (6.6) is a function of λ and θ, while the right-hand side is a function of r only. Therefore, both sides should be equal to a constant, say k.

If we now consider the right-hand side of (6.6), we obtain

$$-\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=k,$$

$$k+\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=0,$$

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+kR=0. \tag{6.7}$$

Source: Krishnamurti et al. 2006

Let the solution of (6.7) be of the form $R = r^n$, where n is an integer. If n is positive, then on substitution of $R = r^n$ into (6.7) we obtain

$$\frac{d}{dr}\left(r^2 \frac{dr^n}{dr}\right) + r^n k = 0\,,$$

$$\frac{d}{dr}\left(r^2 n r^{n-1}\right) + r^n k = 0\,,$$

$$n\frac{d}{dr} r^{n+1} + r^n k = 0\,,$$

$$n(n+1)r^n + r^n k = 0\,,$$

$$\left[n(n+1) + k\right]R = 0\,, \text{ or}$$

$$k = -n(n+1)\,.$$

We see that the value of k is not altered if we replace n by $-n-1$. Therefore r^{-n-1} is also a solution of (6.7), for which $k = -n(n-1)$. Thus k is of the form $-n(n+1)$, n being a positive integer including zero. Equation (6.6) thus reduces to

$$\frac{1}{P\cos\theta}\frac{d}{d\theta}\left(\cos\theta\frac{dP}{d\theta}\right) + \frac{1}{L\cos^2\theta}\frac{d^2L}{d\lambda^2} = -n(n+1)\,,$$

or multiplying by $\cos^2\theta$ and rearranging we obtain

$$\frac{\cos\theta}{P}\frac{d}{d\theta}\left(\cos\theta\frac{dP}{d\theta}\right) + n(n+1)\cos^2\theta = -\frac{1}{L}\frac{d^2L}{d\lambda^2}\,. \qquad (6.8)$$

Source: Krishnamurti et al. 2006

Since the left-hand side of (6.8) is a function of θ and the right-hand side is a function of λ, both sides must be equal to a constant, say m^2. Considering the right-hand side of (6.8) gives

$$-\frac{1}{L}\frac{d^2L}{d\lambda^2} = m^2,$$

$$\frac{d^2L}{d\lambda^2} + m^2L = 0, \qquad (6.9)$$

which has a solution of the form

$$L = e^{\pm im\lambda}, \; m = 0, 1, 2, 3, \ldots$$

Hence after multiplying (6.8) by $P/\cos^2\theta$ and remembering both sides of the equation are equal to m^2 the equation reduces to

$$\frac{1}{\cos\theta}\frac{d}{d\theta}\left(\cos\theta\frac{dP}{d\theta}\right) + n(n+1)P = \frac{m^2P}{\cos^2\theta},$$

$$\frac{1}{\cos\theta}\frac{d}{d\theta}\left(\cos\theta\frac{dP}{d\theta}\right) + \left(n(n+1) - \frac{m^2}{\cos^2\theta}\right)P = 0. \qquad (6.10)$$

Source: Krishnamurti et al. 2006

If $\mu = \sin\theta$ and $\frac{d}{d\theta} = \frac{d}{d\mu}\frac{d\mu}{d\theta} = \frac{d}{d\mu}\frac{d\sin\theta}{d\theta} = \cos\theta\frac{d}{d\mu}$, then (6.10) can be written as

$$\frac{d}{d\mu}\left(\left(1-\mu^2\right)\frac{dP}{d\mu}\right)+\left(n\left(n+1\right)-\frac{m^2}{1-\mu^2}\right)P=0, \qquad (6.11)$$

remembering $1-\sin^2\theta = \cos^2\theta$. Since θ varies from -π/2 to π/2, μ varies from -1 to 1. Equations (6.10) and (6.11) are both known as *associated Legendre equations*, as they apply to all wavenumbers, m.

6.2 Legendre Equation and Associated Legendre Equation

If $m = 0$ in equation (6.11), we obtain

$$\frac{d}{d\mu}\left(\left(1-\mu^2\right)\frac{dP}{d\mu}\right)+n(n+1)P=0, \qquad (6.12)$$

which is called the *Legendre equation*. We next discuss the solutions of the Legendre equation and the associated Legendre equation. Furthermore, we discuss the properties of their solutions without rigorous mathematical proofs.

Source: Krishnamurti et al. 2006

Solutions of the Legendre equation are known as *Legendre polynomials* and are denoted by $P_n(\mu)$. For a given n, $P_n(\mu)$ is a polynomial of degree n and is given by

$$P_n(\mu) = \sum_{r=0}^{M} (-1)^r \frac{(2n-2r)!}{2^n r!(n-r)!(n-2r)!} \mu^{n-2r}, \qquad (6.13)$$

where $M = n/2$ if n is even and $M = (n-1)/2$ if n is odd.

A more convenient form of $P_n(\mu)$ is given by *Rodrigues' formula*, namely

$$P_n(\mu) = \frac{1}{2^n n!} \frac{d^n}{d\mu^n} \left(\mu^2 - 1\right)^n, \; n = 0, 1, 2, 3, \ldots, \; |\mu| \le 1. \qquad (6.14)$$

In particular,

$$P_0(\mu) = 1,$$

$$P_1(\mu) = \frac{1}{2} \frac{d}{d\mu} \left(\mu^2 - 1\right) = \mu,$$

$$P_2(\mu) = \frac{1}{8} \frac{d^2}{d\mu^2} \left(\mu^2 - 1\right)^2 = \frac{1}{2} \left(3\mu^2 - 1\right),$$

$$P_3(\mu) = \frac{1}{48} \frac{d^3}{d\mu^3} \left(\mu^2 - 1\right)^3 = \frac{1}{2} \left(5\mu^2 - 3\mu\right),$$

$$P_4(\mu) = \frac{1}{384} \frac{d^4}{d\mu^4} \left(\mu^2 - 1\right)^4 = \frac{1}{8} \left(35\mu^4 - 30\mu^2 + 3\right),$$

$$P_5(\mu) = \frac{1}{3840} \frac{d^5}{d\mu^5} \left(\mu^2 - 1\right)^5 = \frac{1}{8} \left(63\mu^5 - 70\mu^3 + 15\mu\right).$$

Source: Krishnamurti et al. 2006

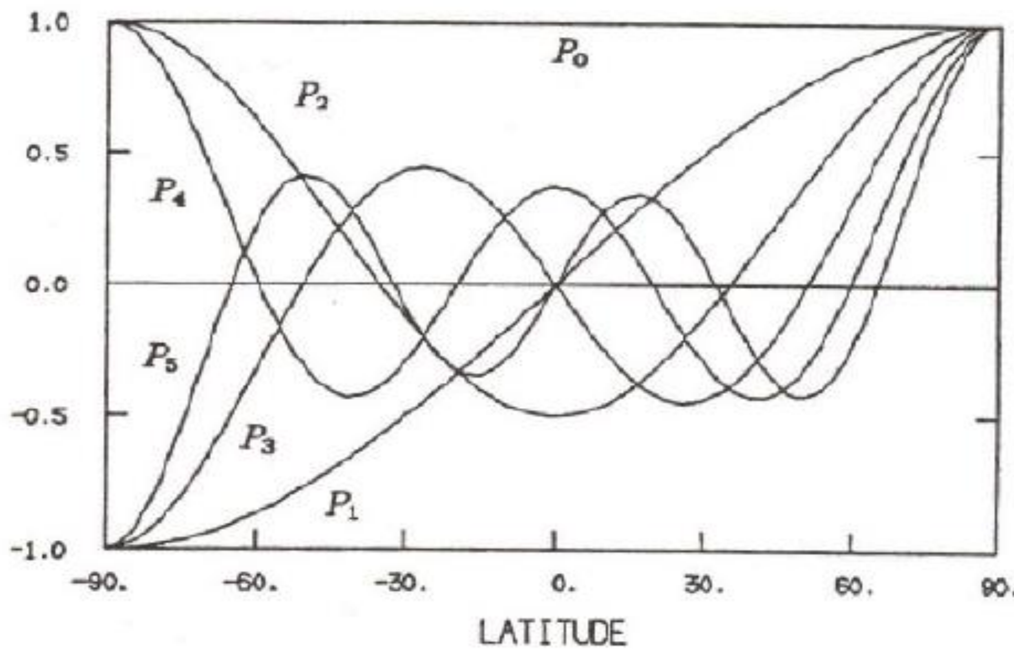


Figure 6.1. Representation of Legendre polynomials $P_0(\mu)$ to $P_5(\mu)$. The number of zero crossings is the same as the degree of the polynomial, n. If n is even, $P_n(\mu)$ is symmetric about the equator. If n is odd, $P_n(\mu)$ is antisymmetric about the equator.

Figure 6.1 shows the graphs of $P_0(\mu)$ to $P_5(\mu)$ for $-1 \le \mu \le 1$.

Three useful properties of $P_n(\mu)$ are as follows: (1) $P_n(\mu = 1) = 1$. (2) If n is even, $P_n(\mu)$ has only even powers of μ and is symmetric with respect to the equator (μ = 0). (3) If n is odd, $P_n(\mu)$ has only odd powers of μ and is antisymmetric with respect to the equator. In other words, for odd n, the graph on the negative side of the μ-axis is a mirror image of the graph on the positive side of the μ-axis, as shown in Fig. 6.1.

Legendre Polynomials : Convenient propertie for treating the latitude variation on sphere & hence used in global AGCMs.

$$P_l(\mu) = \frac{1}{2^l\, l!} \frac{d^l}{d\mu^l}\left[(\mu^2-1)^l\right] \qquad l = 0, 1, 2$$

$$\left(\begin{array}{l}\mu = \sin\phi \\ \phi = \text{latitude}\end{array}\right)$$

Properties

$$P_l(1) = 1 \qquad l = 0, 1, 2, \ldots$$

$$P_l(-1) = (-1)^l \qquad l = 0, 1, 2, \ldots$$

$$\int_{-1}^{1} P_l(\mu)\, Q_k(\mu)\, d\mu = 0 \quad \text{for } k < l$$

$Q_k(\mu)$ = Polynomial of degree $k < l$

$$\frac{(2k+1)}{2}\int_{-1}^{1} P_k(\mu)\, P_l(\mu)\, d\mu = 0 \quad \text{if } k \neq l$$
$$= 1 \quad \text{if } k = l$$

ORTHOGONAL PROPERTY

$P_l(\mu)$ have l distinct real roots in the interval $(-1, 1)$

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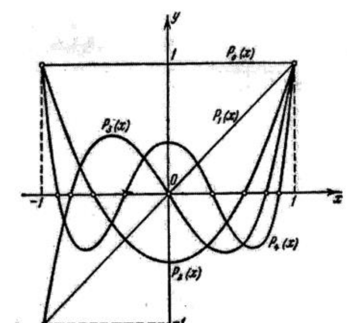
The first five Legendre Polynomials are

$$P_0(\mu) = 1$$
$$P_1(\mu) = \mu$$
$$P_2(\mu) = \frac{1}{2}(3\mu^2 - 1)$$
$$P_3(\mu) = \frac{1}{2}(5\mu^3 - 3\mu)$$
$$P_4(\mu) = \frac{1}{8}(35\mu^4 - 30\mu^2 + 3)$$



The first five Legendre Polynomials $P_l(\mu)$, $l = 0, 1, 2, 3, 4, 5$ plotted as function of (μ) in the interval $(-1, 1)$

Note that $|P_l(\mu)| = 1$ for $\mu = \pm 1$ (ie, at the poles)

So, Legendre Polynomials are very Convenient for Calculations on a sphere

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Legendre Representation

- ***Legendre* Polynomials**

$$p_l(\mu) = \frac{1}{2^l l!} \frac{d^l \left\{(\mu^2 - 1)^l\right\}}{d\mu^l} \qquad l = 0,1,2,\ldots., \qquad |\mu| \le 1$$

- **Associated *Legendre* Polynomials**

$$p_l^k(\mu) = \left(1 - \mu^2\right)^{k/2} \frac{d^k}{d\mu^k} p_l(\mu)$$

Degree = l

Order = k

ASSOCIATED LEGENDRE POLYNOMIALS

$$P_l^m(\mu) = (1-\mu^2)^{m/2} \frac{d^m}{d\mu^m}\left[P_l(\mu)\right]$$

m = Order of Assoc. LEGENDRE POLYNOMIALS
l = Degree of Assoc LEGENDRE POLYNOMIALS

$P_l^m(\mu)$ are solutions of the Laplace Eq^n in spherical co-ordinates

$$(1-\mu^2)\frac{\partial^2}{\partial\mu^2} P_l^m(\mu) - 2\mu\frac{\partial}{\partial\mu} P_l^m(\mu) + \left[l(l+1) - \frac{m^2}{1-\mu^2}\right] P_l^m(\mu) = 0$$

Assoc. LEGENDRE POLYNOMIALS $P_l^m(\mu)$ – are related to spherical harmonics $Y_l^m(\lambda,\mu)$

$$Y_l^m(\lambda,\mu) = P_l^m(\mu)\, e^{im\lambda}$$

Y_l^m are Eigen-functions of the 2-dimensional Laplace equation – (i.e Solutions of the following equation)

$$\nabla^2 Y_l^m + l(l+1) Y_l^m = 0$$

Properties of Assoc. LEGENDRE POLYNOMIALS

$$\int_{-1}^{1} \left[P_l^m(\mu)\right]^2 d\mu = \frac{2}{(2l+1)} \frac{(l+m)!}{(l-m)!}$$

Recursion relationships

$$\epsilon_{l+1}^m P_{l+1}^m = \mu P_l^m - \epsilon_l^m P_{l-1}^m$$

$$(1-\mu^2)\frac{d}{d\mu} P_l^m = -l\,\epsilon_{l+1}^m P_{l+1}^m + (l+1)\,\epsilon_l^m P_{l-1}^m$$

where $\epsilon_l^m = \left[\dfrac{(l^2-m^2)}{(4l^2-1)}\right]^{1/2}$

SPHERICAL HARMONICS $Y_l^m(\lambda,\mu)$ are used in Global AGCMs as Basis Functions

i.e. $$f(\lambda,\mu,t) = a\sum_m \sum_l f_l^m(t)\, Y_l^m(\lambda,\mu)$$

$$= a\sum_m \sum_l f_l^m(t)\, P_l^m(\mu)\, e^{im\lambda}$$

Fields are separated into Fourier Components along Zonal direction and Assoc. LEGENDRE POLYNOMIALS along Meridional direction.

We now consider the associated Legendre equation (6.11). Solutions of this equation involve two parameters, m and n, and are denoted by $P_n^m(\mu)$. $P_n^m(\mu)$ are called *associated Legendre functions of the first kind of order m and degree n*. Here m is any integer and n is a non-negative integer such that $n \geq |m|$.

One can obtain $P_n^m(\mu)$ by using Rodrigues' formula,

$$P_n^m(\mu) = \frac{(1-\mu^2)^{m/2}}{2^n n!} \frac{d^{n+m}}{d\mu^{n+m}} (\mu^2 - 1)^n, \qquad |\mu| \leq 1. \tag{6.15}$$

In particular, for $n = 5$,

$$P_5^0(\mu) = \frac{1}{8}\left(63\mu^5 - 70\mu^3 + 15\mu\right),$$

$$P_5^1(\mu) = \frac{15}{8}\left(1-\mu^2\right)^{1/2}\left(21\mu^4 - 14\mu^2 + 1\right),$$

$$P_5^2(\mu) = \frac{105}{2}\left(1-\mu^2\right)\left(3\mu^3 - \mu\right),$$

Source: Krishnamurti et al. 2006

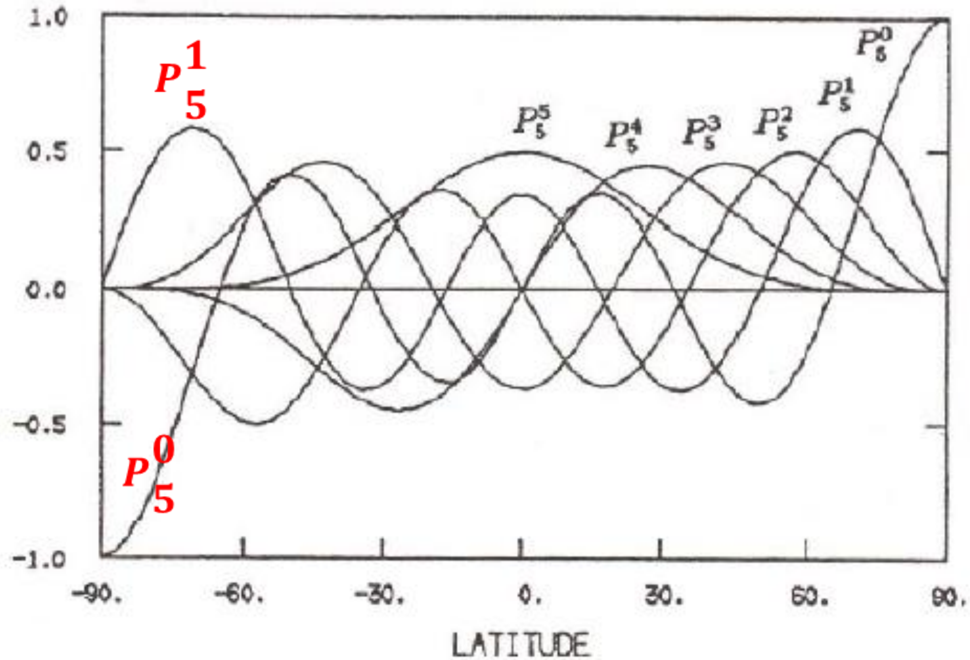


Figure 6.2. Representation of associated Legendre functions $P_5^0(\mu)$ to $P_5^5(\mu)$. Each $P_n^m(\mu)$ is normalized by multiplying by $(n-m)!/(n+m)!$. If $n-m$ is even, $P_n^m(\mu)$ is symmetric about the equator and is antisymmetric if $n-m$ is odd.

$$P_5^3(\mu) = \frac{105}{2}\left(1-\mu^2\right)^{3/2}\left(9\mu^2-1\right),$$

$$P_5^4(\mu) = 945\left(1-\mu^2\right)^2\mu,$$

$$P_5^5(\mu) = 945\left(1-\mu^2\right)^{5/2}.$$

Figure 6.2 shows the graphical representation of the above associated Legendre functions.

Since $P_n^m(\mu)$ is a polynomial of degree n, it has n roots given by the equation $P_n^m(\mu)=0$. It is clear from (6.15) that m of these roots are at the poles ($\mu=\pm 1$), while $n-m$ roots are between the poles. The $n-m$ roots between the poles are called the zeroes of the *associated Legendre function* $P_n^m(\mu)$. If $n-m$ is even, then $P_n^m(\mu)$ is symmetric with respect to the equator. If $n-m$ is odd, then $P_n^m(\mu)$ is antisymmetric with respect to the equator. Three useful properties of $P_n^m(\mu)$ are as follows:

$$P_n^m(\mu)=0 \qquad \text{if } n<m,$$

$$P_n^{-m}(\mu)=(-1)^m\frac{(n-m)!}{(n+m)!}P_n^m(\mu),$$

$$P_n^m(-\mu)=(-1)^{n-m}P_n^m(\mu).$$

6.3 Laplace's Equation

So far we have been working with only one independent variable. We next consider two independent variables. Laplace's equation can be written as

$$\frac{d}{d\mu}\left((1-\mu^2)\frac{dY}{d\mu}\right)+\frac{1}{1-\mu^2}\frac{\partial^2 Y}{\partial\lambda^2}+n(n+1)Y=0, \qquad (6.16)$$

where λ is longitude, $\mu=\sin\theta$, with θ being latitude.

Source: Krishnamurti et al. 2006

Equation (6.16) is of the form $\nabla^2 Y + n(n+1)Y = 0$, where ∇^2 is the two-dimensional Laplacian on a sphere. Let the solution of (6.16) be of the form $Y(\mu,\lambda) = P(\mu)L(\lambda)$. Substituting this into (6.16), we obtain

$$\boxed{\frac{d}{d\mu}\left((1-\mu^2)\frac{dP}{d\mu}\right)} L + \frac{P}{1-\mu^2}\frac{\partial^2 L}{\partial \lambda^2} + n(n+1)PL = 0, \tag{6.17}$$

After dividing by PL and multiplying by $1-\mu^2$, (6.17) may be written as

$$\frac{1-\mu^2}{P}\frac{d}{d\mu}\left((1-\mu^2)\frac{dP}{d\mu}\right) + n(n+1)\left(1-\mu^2\right) = \frac{-1}{L}\frac{d^2 L}{d\lambda^2}. \tag{6.18}$$

The left-hand side of the above equation is a function of μ, while the right-hand side is a function of λ. This implies that both sides must be equal to some constant. Let this constant be m^2, so that from the right-hand side of (6.18) we obtain

$$-\frac{1}{L}\frac{d^2 L}{d\lambda^2} = m^2,$$

$$\frac{d^2}{d\lambda^2} = -Lm^2$$

$$\frac{d^2 L}{d\lambda^2} + m^2 L = 0. \tag{6.19}$$

Source: Krishnamurti et al. 2006

The solution of (6.19) is given by $L = e^{\pm im\lambda}$. Furthermore, from the left-hand side of (6.18) we obtain

$$\frac{1-\mu^2}{P}\frac{d}{d\mu}\left((1-\mu^2)\frac{dP}{d\mu}\right)+n(n+1)(1-\mu^2)=m^2,$$

or upon multiplication by $P/(1-\mu^2)$

$$\frac{d}{d\mu}\left((1-\mu^2)\frac{dP}{d\mu}\right)+\left(n(n+1)-\frac{m^2}{1-\mu^2}\right)P=0, \qquad (6.20)$$

which is an associated Legendre equation. As mentioned in Section 6.2, this has a solution of the form $P_n^m(\mu)$.

Thus given the solution of equation (6.16), $Y(\mu,\lambda)=P(\mu)L(\lambda)$, we found that $L=e^{\pm im\lambda}$ using equation (6.19) and that $P(\mu)$ has the form $P_n^m(\mu)$ using equation (6.20). Hence, the solution of Laplace's equation on a sphere is of the form $Y_n^m(\mu,\lambda)=P_n^m(\mu)e^{im\lambda}$. $Y_n^m(\mu,\lambda)$ is a *spherical harmonic* of order m and degree n. The factor $e^{im\lambda}$ describes the east-west variation, and the factor $P_n^m(\mu)$ describes the north-south variation of the spherical harmonic wave $Y_n^m(\mu,\lambda)$. Some useful mathematical south variation of the spherical harmonic wave $Y_n^m(\mu,\lambda)$. Some useful mathematical properties of $Y_n^m(\mu,\lambda)$ are

$$Y_n^m(\mu,\lambda)=0 \qquad \text{for } n<m,$$

$$Y_n^{m*}(\mu,\lambda)=P_n^m(\mu)e^{-im\lambda},$$

$$Y_n^{-m}(\mu,\lambda)=P_n^{-m}(\mu)e^{-im\lambda}=(-1)^m\frac{(n-m)!}{(n+m)!}P_n^m(\mu)e^{-im\lambda},$$

$$\nabla^2 Y_n^m=\frac{-n(n+1)}{a^2}Y_n^m,$$

where $Y_n^{m*}(\mu,\lambda)$ is the complex conjugate of $Y_n^m(\mu,\lambda)$ and

$$\nabla^2=\frac{1}{a^2\cos\theta}\left(\frac{1}{\cos\theta}\frac{\partial^2}{\partial\lambda^2}+\frac{\partial}{\partial\theta}\cos\theta\frac{\partial}{\partial\theta}\right),$$

and a is the radius of the sphere.

Source: Krishnamurti et al. 2006

6.4 Orthogonality Properties

The spectral equations that we work with contain a system of nonlinear differential equations that are functions of time only. In order to derive such a set of equations, one needs to remove the spatial dependence. Spherical harmonics (containing trigonometric functions in the zonal direction and associated Legendre functions in the meridional direction) describe this spatial dependence. These are removed by invoking orthogonality properties of the trigonometric and Legendre functions using what are called *Fourier* and *Legendre transforms*. In this section, we provide a theoretical background for the desired orthogonality properties.

Legendre polynomials satisfy the orthogonality condition

$$\int_{-1}^{1} P_m(\mu) P_n(\mu)\, d\mu = \begin{cases} 0 & \text{if } m \neq n \\ \dfrac{2}{2n+1} & \text{if } m = n \end{cases}. \tag{6.21}$$

To prove this, we first evaluate the integral

$$\int_{-1}^{1} \mu^m P_n(\mu)\, d\mu, \qquad m \leq n. \tag{6.22}$$

Using Rodrigues' formula and integrating the above equation by parts m times, we have

For details about the proof, please see Krishnamurti et al. 2006

In practice, for spectral modeling it is more convenient to use normalized Legendre polynomials and associated Legendre functions. We may normalize a Legendre polynomial $P_n(\mu)$ by multiplying it by $\left[(2n+1)/2\right]^{1/2}$. Denoting a normalized polynomial by $\tilde{P}_n(\mu)$, we then write the orthogonality relation as

$$\int_{-1}^{1} \tilde{P}_m(\mu)\,\tilde{P}_n(\mu) = \begin{cases} 0 \text{ if } m \neq n \\ 1 \text{ if } m = n \end{cases}. \tag{6.35}$$

Likewise, a normalized associated Legendre function $\tilde{P}_n^m(\mu)$ is given by

$$\tilde{P}_n^m(\mu) = \left(\frac{(n-m)!}{(n+m)!}\right)^{1/2} \left(\frac{(2n+1)}{2}\right)^{1/2} P_n^m(\mu),$$

and the corresponding orthogonality relation becomes

$$\int_{-1}^{1} \tilde{P}_{n_1}^{m_1}(\mu)\,\tilde{P}_{n_2}^{m_2}\,d(\mu) = \begin{cases} 0 \text{ if } m_1 \neq m_2 \text{ and/or } n_1 \neq n_2 \\ 1 \text{ if } m_1 = m_2 = m \text{ and } n_1 = n_2 = n \end{cases}. \tag{6.36}$$

Note that with this normalization, $\tilde{P}_n^{-m}(\mu) = (-1)^m\,\tilde{P}_n^m$.

Throughout the remainder of the book, we deal with normalized Legendre polynomials and normalized associated Legendre functions. We will, however, drop the tilde (˜) so that from now on $P_n(\mu)$ and $P_n^m(\mu)$ represent normalized Legendre polynomial and normalized associated Legendre function, respectively, unless otherwise stated.

Source: Krishnamurti et al. 2006

6.5 Recurrence Relations

Taking a derivative in the east-west direction of a spherical harmonic given by $Y_n^m = P_n^m e^{im\lambda}$ we obtain

$$\frac{\partial}{\partial\lambda} Y_n^m = \frac{\partial}{\partial\lambda} P_n^m e^{im\lambda}$$
$$\frac{\partial}{\partial\lambda} Y_n^m = imP_n^m e^{im\lambda} .$$

However, taking a derivative in the north-south direction of the same harmonic is more complicated. For this we need recurrence relations. There are a number of relations relating the associated Legendre functions of different orders and degrees as well as their derivatives. These are useful for calculating associated Legendre functions and their derivatives needed for spectral modeling. For our purpose, the following *four recurrence relations* for the normalized associated Legendre functions are most useful:

(1) $\mu P_n^m(\mu) = \in_{n+1}^m P_{n+1}^m(\mu) + \in_n^m P_{n-1}^m(\mu)$, or

$$\sin\theta P_n^m(\sin\theta) = \in_{n+1}^m P_{n+1}^m(\sin\theta) + \in_n^m P_{n-1}^m(\sin\theta) , \tag{6.37}$$

where

$$\in_n^m = \left(\frac{n^2 - m^2}{4n^2 - 1}\right)^{1/2} .$$

(2) $\left(1-\mu^2\right) dP_n^m(\mu)/d\mu = -n \in_{n+1}^m P_{n+1}^m(\mu) + (n+1) \in_n^m P_{n-1}^m(\mu)$, or

$$\cos\theta \frac{dP_n^m(\sin\theta)}{d\theta} = -n \in_{n+1}^m P_{n+1}^m(\sin\theta) + (n+1) \in_n^m P_{n-1}^m(\sin\theta) . \tag{6.38}$$

Eliminating P_{n+1}^m between (6.37) and (6.38), we obtain the following relations:

(3) $\left(1-\mu^2\right)dP_n^m(\mu)/d\mu=(2n+1)\in_n^m P_{n-1}^m(\mu)-n\mu P_n^m(\mu)$, or

$$\cos\theta\frac{dP_n^m(\sin\theta)}{d\theta}=(2n+1)\in_n^m P_{n-1}^m(\sin\theta)-n\sin\theta P_n^m(\sin\theta). \qquad (6.39)$$

(4) $\left(1-\mu^2\right)^{1/2}P_n^m(\mu)=g_n^m P_{n+1}^{m+1}(\mu)-h_n^m P_{n-1}^{m+1}(\mu)$, or

$$\cos\theta P_n^m(\sin\theta)=g_n^m P_{n+1}^{m+1}(\sin\theta)-h_n^m P_{n-1}^{m+1}(\sin\theta), \qquad (6.40)$$

where

$$g_n^m=\left(\frac{(n+m+1)(n+m+2)}{(2n+1)(2n+3)}\right)^{1/2}$$

and

$$h_n^m=\left(\frac{(n-m-1)(n-m)}{(2n+1)(2n-1)}\right)^{1/2}.$$

Using (6.37) to (6.40), we can calculate $P_n^m(\mu)$ and their derivatives for any given m and n. Starting with the value of $P_0^0(\mu)$, the recurrence relations (1) and (4) given by (6.37) and (6.40) can generate the values of associated Legendre functions of any given order m and degree n. The value of normalized $P_0^0(\mu)$ for the global domain (μ = -1 to μ = 1) is $1/(2^{1/2})$, and for the hemispheric domain (μ = -1 to μ = 0 or μ = 0 to μ = 1) it is 1.

Source: Krishnamurti et al. 2006

With $m = 0$ and $n = 0$, recurrence relation (4) becomes

$$g_0^0 P_1^1(\mu) = \left(1-\mu^2\right)^{1/2} P_0^0(\mu). \qquad (6.41)$$

From the above equality, if we know $P_0^0(\mu)$, then we can determine $P_1^1(\mu)$. If we know $P_1^1(\mu)$, then the same recurrence relation can be used to obtain $P_2^2(\mu)$. Thus proceeding recursively, we can find the value of $P_m^m(\mu)$ for any given *m*. Likewise if we know $P_m^m(\mu)$, then recurrence relation (1) determines the value of $P_{m+1}^m(\mu)$. From $P_m^m(\mu)$, and $P_{m+1}^m(\mu)$, the same relation gives value of $P_{m+2}^m(\mu)$. Thus proceeding recursively, we can calculate the value of $P_n^m(\mu)$ for any given degree *n*.

Recurrence relation (2) can be used to calculate the differential of any $P_m^m(\mu)$ from the values of $P_{n-1}^m(\mu)$ and $P_{n+1}^m(\mu)$. Recurrence relation (3) serves a similar purpose, but we need $P_{n-1}^m(\mu)$ and $P_n^m(\mu)$ to calculate differentials of $P_n^m(\mu)$. To calculate the derivative of $P_n^m(\mu)$ of the highest degree *n* using recurrence relation (2), we need the value of $P_{n+1}^m(\mu)$, which is beyond the usual truncation of the series. Recurrence relation (3) uses the values of associated Legendre functions within the truncation limit only. For calculation of differentials of $P_n^m(\mu)$, it is therefore preferable to use recurrence relation (3) rather relation (2).

Source: Krishnamurti et al. 2006

6.6 Gaussian Quadrature

To obtain the coefficients of the truncated spherical harmonics in the spectral model we have to perform a Fourier transform as well as evaluate the Legendre transform. This requires the evaluation of numerical integrals. The *Gaussian quadrature* is used for the Legendre transform of data in the north-south direction. We show that the Gaussian quadrature is an efficient numerical quadrature for this purpose. Furthermore, we describe the procedure to design it.

Consider the integral $\int_a^b f(x)dx$, where $f(x)$ is an integrable function on $a \le x \le b$. We can write this as

$$\int_a^b f(x)dx = \sum_{i=1}^{n} w_i f(x_i), \tag{6.42}$$

where the expression on the right-hand side is the numerical equivalent of the integral on the left-hand side. The expression on the right-hand side is known as a *numerical integral quadrature*. Our aim is to select w_i and x_i such that the summation on the right-hand side is exactly equal to the integral on the left-hand side. If we choose x_i equally spaced within the interval of integration, then we shall have n values of w_i as a function of the location of these points.

It is possible to make the quadrature $\sum_{i=1}^{n} w_i f(x_i)$ exact for $f(x)$ of degree $\le n - 1$ by suitable selection of w_i. However, we show that if we can choose both w_i and x_i suitably, it is possible to make the numerical quadrature $\sum_{i=1}^{n} w_i f(x_i)$ exactly equal to the integral on the left-hand side for $f(x)$ of degree $\le 2n - 1$, which is the highest possible accuracy attainable from $2n$ degrees of freedom (n for x_i and n for w_i). We call w_i the *Gaussian weights* and x_i the *Gaussian ordinates*, and the numerical quadrature is called the *Gaussian quadrature*.

It can be shown that x_i are the roots of $P_n(x)$ (i.e., of the Legendre polynomial of degree n). Also,

$$w_i = \frac{2(1-x_i^2)(n-\frac{1}{2})}{n^2 P_{n-1}(x_i)^2}, \tag{6.72}$$

which is a more convenient expression for the Gaussian weights w_i.

Thus if a polynomial $F(x)$ of degree $2n-1$ has values $F(x_i)$ available at n points x_i, $i=1, 2, 3, \ldots, n$, as the roots (zeroes) of the Legendre polynomial P_n, then the integral $\int_{-1}^{1} F(x)dx$ is evaluated exactly by the Gaussian quadrature $\sum_{i=1}^{n} w_i F(x_i)$ where x_i are the zeroes of the Legendre polynomial of degree n and w_i are the weights given by (6.72).

The efficiency of the Gaussian quadrature lies in the fact that it can evaluate the integral of a function of degree $2n-1$ from its values given at n points. For designing a Gaussian quadrature, we need to take the following steps:

Source: Krishnamurti et al. 2006

1. If N is the highest degree of the polynomial to be integrated, then the minimum number of points n needed for the Gaussian quadrature is given by $2n-1=N$, or $n=(N+1)/2$.

2. The Gaussian points x_i, $i=1, 2, \ldots, n$, are determined as the zeroes of the Legendre polynomials of degree n, that is, as the roots of $P_n(x)=0$. The roots of $P_n(x)$ are not equally spaced, but are nearly so. They can be determined iteratively using the Newton-Raphson method starting with the first guess of x_i as n equally spaced points between -1 and 1.

3. Once x_i are found, the weights w_i can be obtained from (6.72).

In the case of a Legendre polynomial of even degree, its zeroes are located symmetrically, in other words, at $\pm x_i$. In the case of a Legendre polynomial of odd degree, $x=0$ is one of the zeroes and the rest of the zeroes would be located symmetrically at $\pm x_i$. The weight w_i has the same value for $\pm x_i$. It is therefore sufficient to calculate the zeroes of the Legendre polynomials and the corresponding weights for $x \geq 0$ to complete the Gaussian quadrature. In a spectral model, the use of a suitable fast Fourier transform (FFT) for the Fourier transform and Gaussian quadrature for the Legendre transform achieves the optimally fast and accurate calculations.

Source: Krishnamurti et al. 2006

For computing spectral coefficients (i.e., coefficients of spherical harmonics), we compute integrals in the north-south direction using the Gaussian Quadrature.

$$\int_{-1}^{1} f(\mu)\, d\mu = \sum_{i=1}^{l} A_i f(\mu_i)$$

$f(\mu)$ = Polynomial of degree $\leq (2l-1)$

$A_i \quad (i = 1, 2 \ldots l) \rightarrow$ Gaussian Weights

$\mu_i \quad (i = 1, 2 \ldots l) \rightarrow$ Gaussian Latitudes

A_i and μ_i are determined using the GAUSSIAN QUADRATURE

It can be shown that

$$A_i = \frac{2(1-\mu_i)^2 (l - \frac{1}{2})}{l^2 P_{l-1}(\mu_i)^2}$$

l is the number of Gaussian Latitudes

μ_i = Roots of the Legendre Polynomial

l roots

$P_l(\mu_i) = 0 \quad (\mu_1, \mu_2, \mu_3 \ldots \mu_l)$

Spectral transform technique

Atmospheric variables can be transformed from grid domain to spectral domain and vice-versa. For example, a dynamical variable on a lat-lon grid (Gaussian grid) is transformed to spectral coefficients as follows

Fourier harmonics are evaluated at each Gaussian latitude through FFT. Here N_{long} are the number of longitudes.

$$f(\lambda_i, \phi_j) \rightarrow f_k(\phi_j) = \frac{1}{N_{long}} \sum_{i=1}^{N_{long}} f(\lambda_i, \phi_j) e^{-ik\lambda_k}$$

The spectral coefficients are finally obtained via the inverse Legendre transformation at of the Gaussian latitudes – using the Gaussian Quadrature. Here N_{lat} correspond the number of latitude circles and w_j are the Gaussian Weights

$$f_k \longrightarrow f_l{}^k = \frac{1}{2} \sum_{j=1}^{N_{lat}} \omega_j \, f_k{}^* P_l{}^k$$

6.7 Spectral Representation of Physical Fields

Any smooth function over a sphere can be expressed as a sum of spherical harmonics. However, the convergence of term-by-term derivatives of the function is assured for much less liberal conditions, that is, the absolute convergence of the series. With most meteorological quantities (scalars), this is generally not a problem. However, sharp discontinuities across clouds, rain areas, fronts, and so on, are not easy to represent. There is also a problem with the horizontal wind components u and v, which have a singular behavior at the poles. While u and v can be expressed to any desired accuracy by a series of spherical harmonics, there is no guarantee that the derivative of such a series will converge properly (this is the pole problem). The cross-polar flow cannot be expressed by a series. At one side, the flow may appear to be northerly, while at the other side of the pole the flow might appear southerly. This poses a singularity at the pole.

In the past u and v were set equal to zero at the poles for all times. However, due to this singularity at the poles, the velocity components are carried in terms of the following pseudoscalar fields (called *Robert functions*):

$$U = \frac{u\cos\theta}{a} \qquad \text{and} \qquad V = \frac{v\cos\theta}{a}, \tag{6.73}$$

where θ is the latitude and a is the radius of the earth. At the poles, $\theta = \pm\pi/2$, so $U = V = 0$, but we can find U and V very close to 90°N and 90°S. The Robert functions U and V are therefore well defined at the poles. It is these functions rather than u and v that we work with.

Source: Krishnamurti et al. 2006

One can obtain a simple kinematic relationship between the flow-field variables in the spectral domain using the formula discussed in earlier sections. We deal with variables such as ψ, χ, ζ, D, U and V, where ψ is the stream function, χ is the velocity potential, ζ is the relative vorticity, D is the divergence, and U and V are the Robert functions defined in (6.73). The relationships between U, V and ψ, χ (or ζ, D) are given by

$$U = \frac{1}{a^2}\left(\frac{\partial \chi}{\partial \lambda} - \cos\theta \frac{\partial \psi}{\partial \theta}\right), \tag{6.74}$$

$$V = \frac{1}{a^2}\left(\frac{\partial \psi}{\partial \lambda} + \cos\theta \frac{\partial \chi}{\partial \theta}\right), \tag{6.75}$$

$$\zeta = \frac{1}{\cos^2\theta}\left(\frac{\partial V}{\partial \lambda} - \cos\theta \frac{\partial \mathrm{U}}{\partial \theta}\right), \tag{6.76}$$

$$D = \frac{1}{\cos^2\theta}\left(\frac{\partial U}{\partial \lambda} + \cos\theta \frac{\partial V}{\partial \theta}\right), \tag{6.77}$$

where θ is the latitude, λ is the longitude, and a is the radius of the earth.

Source: Krishnamurti et al. 2006

If ψ and χ are both well-behaved, which generally is the case, then for a given truncation we can expand ψ and χ by the relations

$$\psi = a^2 \sum_m \sum_n \psi_n^m Y_n^m (\lambda, \mu) \tag{6.78}$$

and

$$\chi = a^2 \sum_m \sum_n \chi_n^m Y_n^m (\lambda, \mu) . \tag{6.79}$$

For the same truncation, remembering the mathematical property $\nabla^2 Y_n^m = -\frac{n(n+1)}{a^2} Y_n^m$ given in Sec. 6.3, we can express the vorticity and divergence as

$$\zeta = \nabla^2 \psi = \sum_m \sum_n -n(n+1) \psi_n^m Y_n^m \tag{6.80}$$

and

$$D = \nabla^2 \chi = \sum_m \sum_n -n(n+1) \chi_n^m Y_n^m . \tag{6.81}$$

Using the spectral expansions $\zeta = \sum_m \sum_n \zeta_n^m Y_n^m$ and $D = \sum_m \sum_n D_n^m Y_n^m$ we get the spectral form of ζ and D for one harmonic so that

$$\zeta_n^m = -n(n+1) \psi_n^m \tag{6.82}$$

and

$$D_n^m = -n(n+1) \chi_n^m . \tag{6.83}$$

Source: Krishnamurti et al. 2006

By substituting the spectral representations of ψ (6.78) and χ (6.79) into (6.74) and (6.75) and making use of the zonal meridional differentials of $Y_n^m(\lambda,\mu)$ given by

$$Y_n^m(\lambda,\mu) = P_n^m(\mu)e^{im\lambda},$$

$$\frac{\partial Y_n^m(\lambda,\mu)}{\partial\lambda} = imP_n^m(\mu)e^{im\lambda},$$

$$\frac{\partial Y_n^m(\lambda,\mu)}{\partial\lambda} = imY_n^m(\lambda,\mu) \qquad (6.84)$$

and using the recurrence relation (6.38) where $e^{im\lambda}$ has been multiplied through

$$\left(1-\mu^2\right)\frac{\partial Y_n^m(\lambda,\mu)}{\partial\mu} = -n\in_{n+1}^m Y_{n+1}^m(\lambda,\mu)$$
$$+(n+1)\in_n^m Y_{n-1}^m(\lambda,\mu), \qquad (6.85)$$

we obtain the spectral expansions of both U and V. That is,

$$U = \sum_m\sum_n U_n^m Y_n^m = \frac{1}{a^2}\left(a^2\sum_m\sum_n\frac{\partial}{\partial\lambda}\chi_n^m Y_n^m - \cos\theta a^2\sum_m\sum_n\frac{\partial}{\partial\theta}\psi_n^m Y_n^m\right).$$

Remembering $Y_n^m = P_n^m e^{im\lambda}$, $\dfrac{\partial}{\partial\theta} = \cos\theta \dfrac{\partial}{\partial\mu}$, and $\mu = \sin\theta$ and using (6.84), we obtain

$$U = \sum_m \sum_n im\chi_n^m Y_n^m - \cos\theta \sum_m \sum_n \cos\theta \frac{\partial}{\partial\mu} \psi_n^m Y_n^m .$$

Using the identity $\cos^2\theta = 1 - \sin^2\theta = 1 - \mu^2$, we obtain

$$U = \sum_m \sum_n im\chi_n^m Y_n^m - \left(1-\mu^2\right) \sum_m \sum_n \psi_n^m \frac{\partial Y_n^m}{\partial\mu} .$$

Finally, using (6.85) we obtain

$$U = \sum_m \sum_n U_n^m Y_n^m = \sum_m \sum_n im\chi_n^m Y_n^m - \sum_m \sum_n \psi_n^m \left[-n \in_{n+1}^m Y_{n+1}^m + (n+1) \in_n^m Y_{n-1}^m \right]. \tag{6.86}$$

Similarly for V we obtain

$$V = \sum_m \sum_n V_n^m Y_n^m = \sum_m \sum_n im\psi_n^m Y_n^m + \sum_m \sum_n \chi_n^m \left[-n \in_{n+1}^m Y_{n+1}^m + (n+1) \in_n^m Y_{n-1}^m \right]. \tag{6.87}$$

Equating the coefficients of Y_n^m on both sides of the above equations, we obtain

Source: Krishnamurti et al. 2006

$$U_n^m = im\chi_n^m + (n-1)\in_n^m \psi_{n-1}^m - (n+2)\in_{n+1}^m \psi_{n+1}^m, \tag{6.88}$$

$$V_n^m = im\psi_n^m - (n-1)\in_n^m \chi_{n-1}^m + (n+2)\in_{n+1}^m \chi_{n+1}^m. \tag{6.89}$$

Since $\zeta_n^m = -n(n+1)\psi_n^m$ and $D_n^m - (n+1)\chi_n^m$, we may also write

$$n(n+1)U_n^m = -imD_n^m - (n+1)\in_n^m \zeta_{n-1}^m + n\in_{n+1}^m \zeta_{n+1}^m, \tag{6.90}$$

$$n(n+1)V_n^m = -im\zeta_n^m + (n+1)\in_n^m D_{n-1}^m - n\in_{n+1}^m D_{n+1}^m. \tag{6.91}$$

For $m = 0$ and $n = 0$ and remembering $Y_n^m(\lambda, \mu) = 0$ for $n < m$,

$$U_0^0 = -2\in_1^0 \psi_1^0 = \in_1^0 \zeta_1^0, \tag{6.92}$$

$$V_0^0 = -2\in_1^0 \chi_1^0 = \in_1^0 D_1^0. \tag{6.93}$$

At this point, let us ask the following questions: How does kinematics work, given ψ_n^m and χ_n^m? If we are given the spectral coefficients of the stream function and the velocity potential (i.e., given ψ_n^m and χ_n^m), can we obtain ζ_n^m, D_n^m, U_n^m, and V_n^m? Since we know ψ_n^m and χ_n^m for all desired values of n and m, we can use (6.88) and (6.89) to obtain U_n^m and V_n^m, respectively.

Source: Krishnamurti et al. 2006

Using (6.82) and (6.83), one can obtain ζ_n^m and D_n^m, respectively. Furthermore, one can obtain the true velocity components u and v using the inverse Robert functions. That is, equations (6.82), (6.83), and (6.88)-(6.91) are a series of equations that enable one to solve for ψ, χ, ζ, D, U and V given two spectral coefficients. This can be called the triangle problem. Without worrying about details, the orthogonality properties of Legendre polynomials and associated Legendre functions are used for determining the expansion coefficients. This leads to

$$\int_0^{2\pi}\int_{-1}^{1} A Y_n^{m*} d\mu\, d\lambda = \sum_m \sum_n A_n^m \int_0^{2\pi}\int_{-1}^{1} Y_n^m Y_n^{m*} d\mu\, d\lambda, \qquad (6.94)$$

where Y_n^{m*} is the complex conjugate of Y_n^m. We have used the orthogonality of the spherical harmonics, that is,

$$\frac{1}{2\pi}\int_0^{2\pi}\int_{-1}^{1} Y_n^m Y_j^{k*} d\mu\, d\lambda = \delta_{mk}\delta_{nj} = \begin{cases} 0 & \text{if } m \neq j \text{ and/or } n \neq j \\ 1 & \text{if } m = k \text{ and } n = j \end{cases}.$$

Transformation from grid space to spectral space and vice versa is important in spectral modeling. Grid space is represented by $A(\lambda,\mu)$ and is the representation on a weather map. Spectral space is represented by A_n^m and are the amplitudes of the spherical harmonics of $A(\lambda,\mu)$. It is important to learn to switch between grid ($A(\lambda,\mu)$) and spectral space (A_n^m), i.e. to get weather maps from the spectral coefficients. To do this we must utilize the Fourier transform, Legendre transform, inverse Fourier transform, and inverse Legendre transform. We can write a function A as

$$A = \sum_m \sum_n A_n^m Y_n^m ,$$

where $A = A(\lambda, \mu)$ and

$$A_n^m = \frac{1}{2\pi} \int_0^{2\pi} \int_{-1}^{1} A Y_n^{m*} d\mu \, d\lambda . \tag{6.95}$$

It is necessary to truncate the series at some wavenumber. There are two common ways of truncating the series: (1) *Rhomboidal truncation*, which has the form

$$A(\lambda, \mu) = \sum_{m=-N}^{N} \sum_{n=|m|}^{|m|+N} A_n^m Y_n^m (\lambda, \mu) , \tag{6.96}$$

and (2) *triangular truncation*, which has the form

$$A(\lambda, \mu) = \sum_{m=-N}^{N} \sum_{n=|m|}^{N} A_n^m Y_n^m (\lambda, \mu) . \tag{6.97}$$

If the set of spectral coefficients A_n^m is known, then by using either (6.96) or (6.97) the function $A(\lambda, \mu)$ can be defined everywhere over the globe. In practice, it is necessary to evaluate the function at a finite set of grid points. Likewise, the evaluation of the integrals in the Fourier-Legendre transform is based on data at only a finite number of points.

Source: Krishnamurti et al. 2006

6.7.1 *Grid to Spectral Space*

In practice, the evaluation of A_n^m from $A(\lambda,\mu)$ is carried out in two steps.

Step 1: Perform the Fourier transform of the space field along latitudes,

$$A^m(\mu) = \frac{1}{2\pi}\int_0^{2\pi} A(\lambda,\mu)e^{-im\lambda}d\lambda. \tag{6.98}$$

Step 2: Perform the Legendre transform of the Fourier components,

$$A_n^m = \int_{-1}^{1} A^m(\mu)P_n^m(\mu)\,d\mu. \tag{6.99}$$

The Fourier transform is evaluate using the trapezoidal quadrature formula, that is,

$$A^m(\mu_k) = \frac{1}{2M}\sum_{j=0}^{2M-1} A(\lambda_j,\mu_k)e^{-im\lambda_j}, \tag{6.100}$$

where $\lambda_j = [(2\pi)/(2M)]j = (\pi/M)j$. This integration is exact for any function which may be represented by a truncated Fourier series with wavenumbers $\leq 2M-1$. This calculation is very efficiently done by the FFT. The Legendre transform is evaluated using the Gaussian quadrature formula, that is,

Source: Krishnamurti et al. 2006

$$A_n^m = \sum_{k=1}^{K} W(\mu_k) A^m(\mu_k) P_n^m(\mu_k), \tag{6.101}$$

where μ_k are the *Gaussian latitudes* and $W(\mu_k)$ are the *Gaussian weights*, as discussed in Section 6.6. This formula is exact for any polynomial [here $A^m(\mu)P_n^m(\mu)$] of degree $\leq 2K-1$.

6.7.2 *Spectral to Grid Space*

The transformation from spectral to grid space is also achieved similarly in two steps.

Step 1: Perform the reverse Legendre transform,

$$A^m(\mu_k) = \sum_n \phi A_n^m P_n^m(\mu_k). \tag{6.102}$$

Step 2: Perform the reverse Fourier transform,

$$A(\lambda_j, \mu_k) = A_n^m(\mu_k) e^{im\lambda_j}. \tag{6.103}$$

Thus one proceeds from spherical harmonic components A_n^m to Fourier components A^m and then to grid-point values $A(\lambda_j, \mu_k)$. It should be noted that the use of Gaussian latitudes and weights enables one to calculate the Legendre transform exactly.

Source: Krishnamurti et al. 2006

Truncation

a) Rhomboidal truncation

$$f(\lambda,\mu,\sigma,t) = a\sum_{m=-J}^{J}\sum_{l=|m|}^{|m|+J} f_l^m P_l^m(\mu)e^{im\lambda}$$

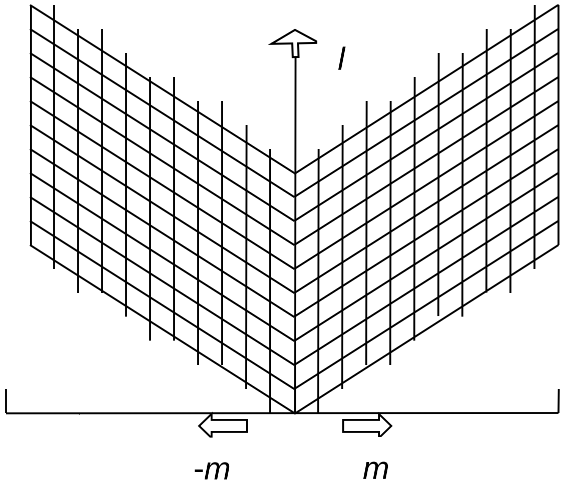


Figure: Rhombodial truncation

b) Triangular Truncation

$$f(\lambda,\mu,\sigma,t) = a\sum_{m=-J}^{J}\sum_{l=|m|}^{J} f_l^m P_l^m(\mu)e^{im\lambda}$$

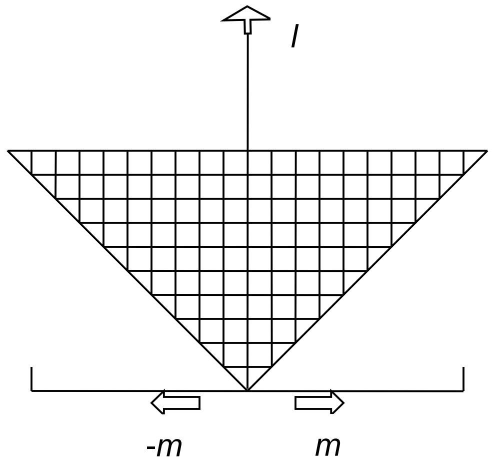


Figure: Triangular truncation

Choice of grid size

- Aliasing : If we have *2N + 1* grid points, we represent them into N waves by analyzing them into a maximum of *N-Fourier* (N-sine and N-cosine) components. If one tries to resolve more than N waves out of 2N +1 data points, a false representation (aliasing) of lower wave numbers into higher wave numbers occurs. So, the number of grid-points should be chosen such that the computations are free from aliasing.

- In addition to alias free computations, the FFT computations require that the number of grid points along the zonal direction satisfy $N = N_o 2^p$ (where N_o = 1 or 3 or 5 and p is positive). For example: for T42 truncation, N=42 thus $N_{\_Nlon} = 3N+1 \leq 128$. Note that $128=2^7$.

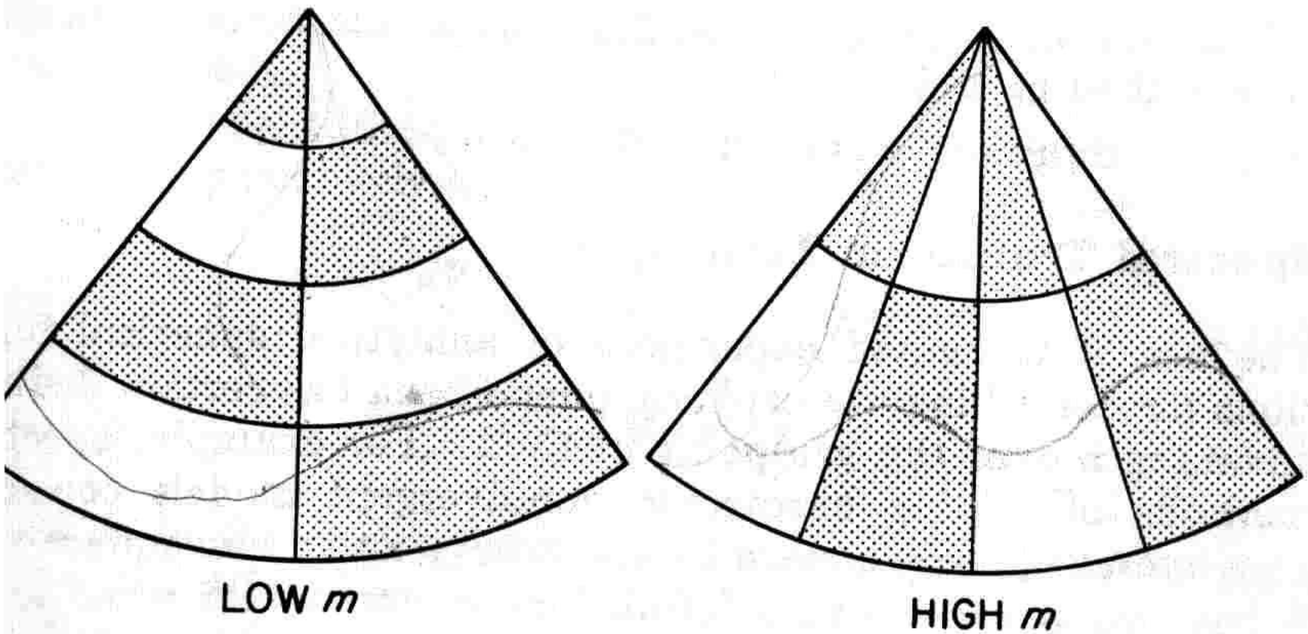
- For *Legendre* transform in the meridional (North-South) direction we need to have more than $N_{\_Nlat} = (3N+1)/2$ grid points in case of triangular truncation and more than $N_{\_Nlat} = (5N+1)/2$ grid points for rhomboidal truncation.

Table I. Truncations for wave number J.

Rhomboidal: No. of Gaussian latitudes ≥ (5J+1)/2 and No. of points per latitude circles > (3J+1)
Total number of spectral coefficients ≤ $2(J+1)^2$.

Triangular: No. of Gaussian latitudes ≥ (3J+1)/2 and No. of points per latitude circles > (3J+1)
Total number of spectral coefficients ≤ (J+1)(J+2).

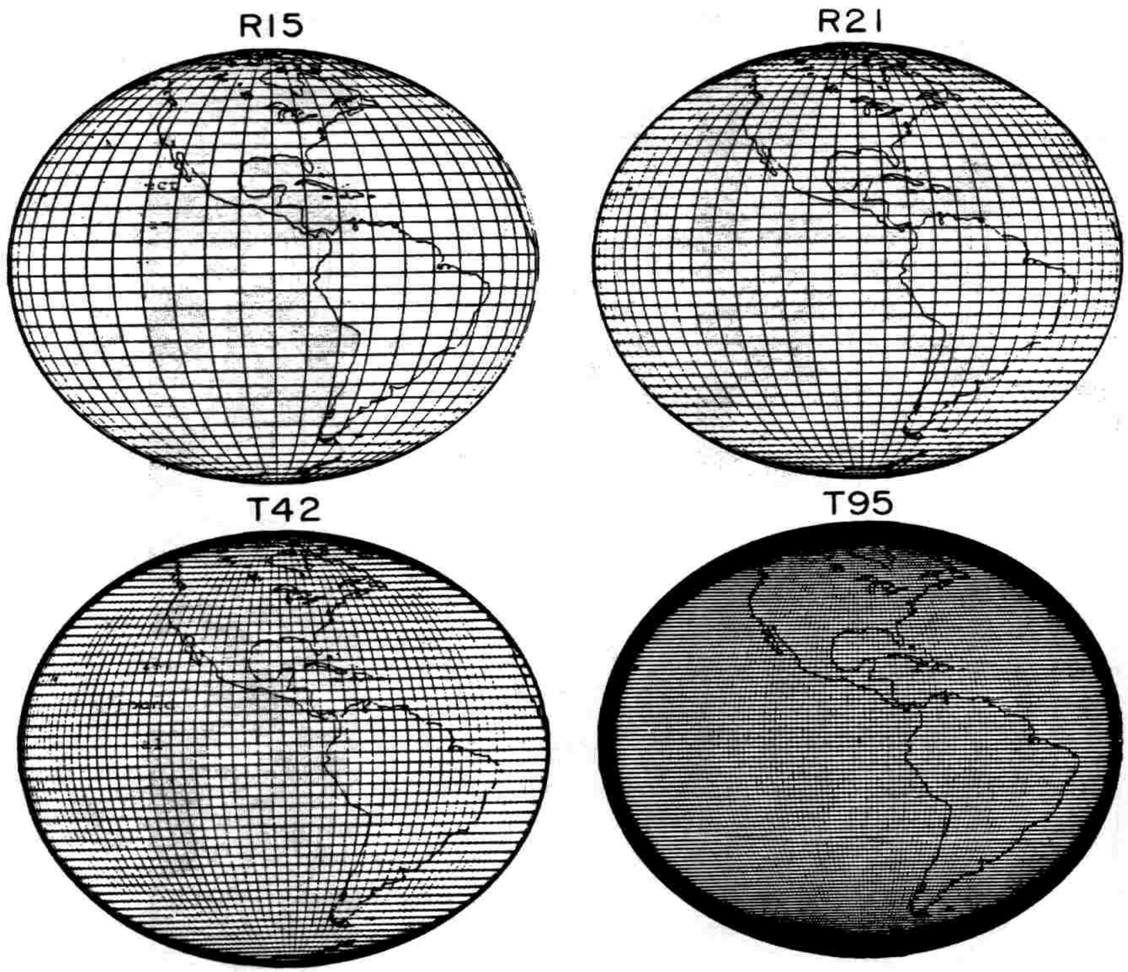
S.No.	Wave number truncation	No. of Gaussion latitudes (latitude circles) over 0-180^O	Grid size in Latitudinal direction in degree	No. of points per latitude circle 0-360^O	Grid size in Longitudinal direction in degree	Total no. of grid points.	Total no. of spectral coefficients
Rhomboidal Truncation							
1	**R 7**	20	9	24	15	480	98
2	**R 12**	32	5.625	48	7.5	1536	288
3	**R 15**	40	4.5	64 (48)	5.5 (7.5)	2560 (1920)	512
4	**R 21**	54	3.3	64	5.5	3456	882
5	**R 24**	62	2.9	80	4.5	4960	1152
6	**R 30**	80	2.25	96	3.75	7680	1922
7	**R 40**	100	1.8	128	2.8	12800	3200
Triangular Truncation							
8	**T 15**	24	7.5	48	7.5	1152	272
9	**T 30**	48	3.75	96	3.75	4608	992
10	**T 40**	64	2.8	128	2.8	8192	1722
11	**T42**	64	2.8	128	2.8	8192	1892
12	**T62**	94	1.9	192	2.5	18048	4032
13	**T 63**	96	1.875	192	1.875	18432	4095
14	**T 80**	128	1.4	256	1.4	32768	6642
15	**T 95**	144	1.25	288	1.25	41472	9312
16	**T 126**	190	0.9	384	0.9	73960	16256



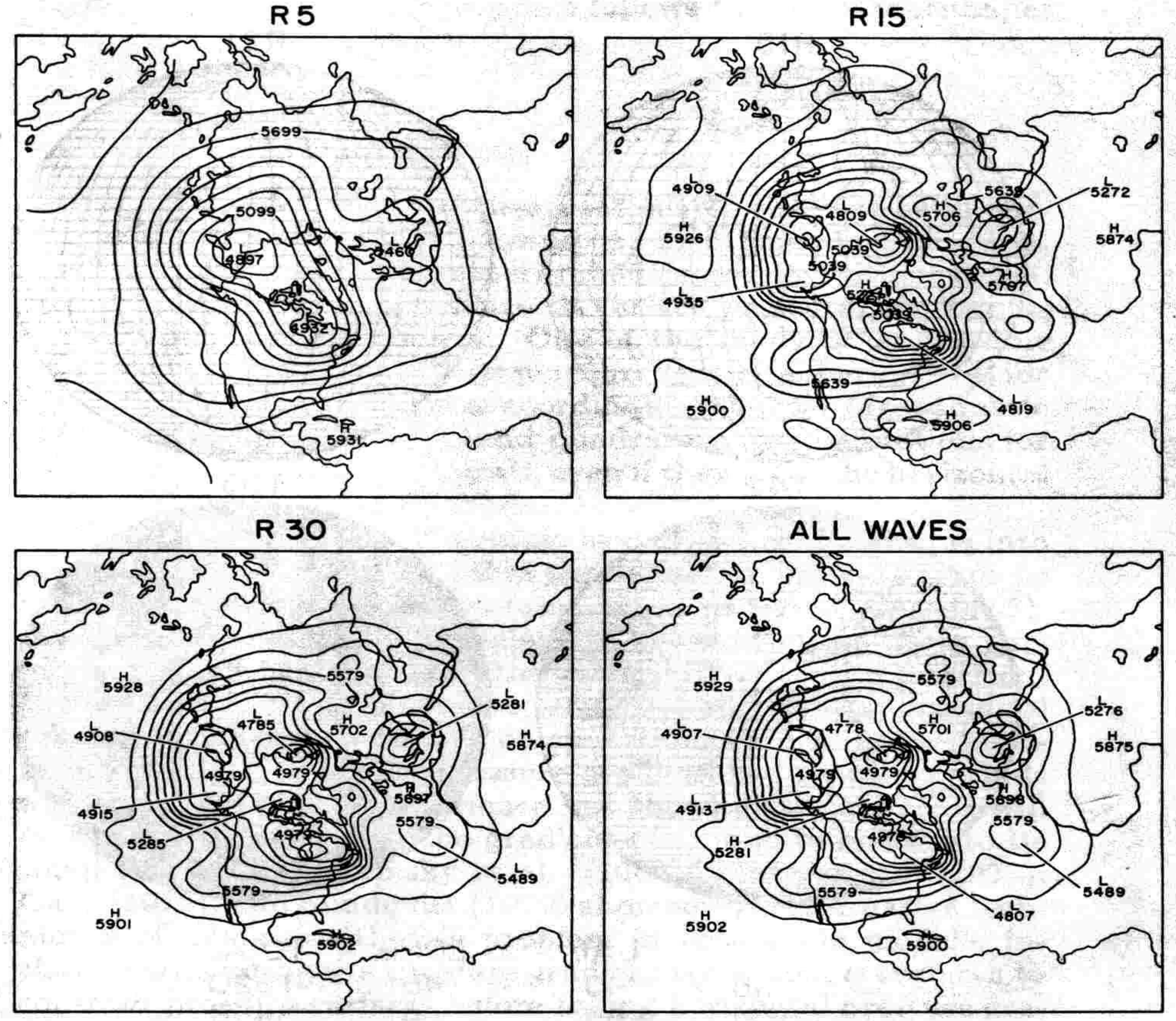
. 4.7 Alternating patterns of positive and negative deviations of the potential from the mean geopotential, gH, for low and high values of planetary wave number m.

Left panel : $m = 1$, $l = 3$

Right panel: $m = 2$, $l = 3$



	LAT.	LONG.	TOTAL
R15	40	48	1920
R21	54	64	3456
T42	64	128	8192
T95	144	288	41472



Effect of various horizontal truncations on the 500 hPa geopotential Height (m) of 2.5 x 2.5 lat-lon data. Rhomboidal R5, R15, R30 and the original data on 2.5^{o} x 2.5^{o} grid (ALL WAVES)

6.8 Barotropic Spectral Model on a Sphere

In Chapter 2 we discussed the finite-difference barotropic model where we were concerned with the proper formulation of the space-differencing schemes, the Jacobian, the Laplacian, and the solution of Poisson's equation. In this section, we consider a spectral model on a sphere, the integration of which will be through a coupled system of nonlinear ordinary differential equations. This system of equations is obtained by transforming the model equations from the space to the spectral domain. The basis functions for this transformation will be surface spherical harmonics, some important properties of which have already been discussed.

If λ represents the longitude and θ the latitude, then the barotropic vorticity equation on a sphere is given by

$$\frac{\partial \zeta}{\partial t} = \frac{1}{a^2}\left(\frac{\partial \psi}{\partial \mu}\frac{\partial}{\partial \lambda}(\zeta + f) - \frac{\partial \psi}{\partial \lambda}\frac{\partial}{\partial \mu}(\zeta + f)\right), \qquad (6.104)$$

where $\mu = \sin\theta$. Noting that $\partial f / \partial \lambda = 0$ and $\partial f / \partial \mu = 2\Omega$, we obtain

$$\frac{\partial \zeta}{\partial t} = \frac{1}{a^2}\left(\frac{\partial \psi}{\partial \mu}\frac{\partial \zeta}{\partial \lambda} - \frac{\partial \psi}{\partial \lambda}\frac{\partial \zeta}{\partial \mu}\right) - \frac{2\Omega}{a^2}\frac{\partial \psi}{\partial \lambda},$$

or

$$\frac{\partial \zeta}{\partial t} = F(\lambda, \mu) - \frac{2\Omega}{a^2}\frac{\partial \psi}{\partial \lambda}. \qquad (6.105)$$

Source: Krishnamurti et al. 2006

In the above equation, the nonlinear advective term

$$\frac{1}{a^2}\left(\frac{\partial\psi}{\partial\mu}\frac{\partial\zeta}{\partial\lambda}-\frac{\partial\psi}{\partial\lambda}\frac{\partial\zeta}{\partial\mu}\right)$$

is written as $F(\lambda,\mu)$. The second term on the right-hand side of (6.105) is the earth's rotation term and it is linear.

To transform (6.105) into its spectral form, we take the Fourier-Legendre expansion of variables ζ, ψ, and F

$$\zeta(\lambda,\mu,t)=\sum_m\sum_n \zeta_n^m(t)Y_n^m(\lambda,\mu), \tag{6.106}$$

$$\psi(\lambda,\mu,t)=\sum_m\sum_n \psi_n^m(t)Y_n^m(\lambda,\mu), \tag{6.107}$$

$$F(\lambda,\mu,t)=\sum_m\sum_n F_n^m(t)Y_n^m(\lambda,\mu). \tag{6.108}$$

Substituting into (6.105) we obtain

$$\sum_m\sum_n \frac{d\zeta_n^m(t)}{dt}Y_n^m(\lambda,\mu)=\sum_m\sum_n\Bigg(F_n^m(t)Y_n^m(\lambda,\mu) -\frac{2\Omega}{a^2}\psi_n^m(t)\frac{\partial}{\partial\lambda}Y_n^m(\lambda,\mu)\Bigg). \tag{6.109}$$

Using $\zeta = \nabla^2\psi$ and looking at the left-hand side of equation (6.105) only, we get

$$\frac{d\zeta_n^m(t)}{dt} Y_n^m(\lambda,\mu) = -\frac{n(n+1)}{a^2}\frac{d\psi_n^m(t)}{dt} Y_n^m(\lambda,\mu), \tag{6.110}$$

remembering $\nabla^2 Y_n^m = \dfrac{-n(n+1)}{a^2} Y_n^m$. Also,

$$\frac{\partial}{\partial\lambda} Y_n^m(\lambda,\mu) = imY_n^m(\lambda,\mu). \tag{6.111}$$

Substituting (6.110) and (6.111) into (6.109) and then equating the coefficients of $Y_n^m(\lambda,\mu)$ on both sides of the above equation, we obtain

$$\frac{-n(n+1)}{a^2}\frac{d\psi_n^m(t)}{dt} Y_n^m(\lambda,\mu) = F_n^m(t) Y_n^m(\lambda,\mu) - \frac{2\Omega}{a^2}\psi_n^m(t) imY_n^m(\lambda,\mu)$$

$$\frac{d\psi_n^m}{dt} = \frac{2\Omega im}{n(n+1)}\psi_n^m - \frac{a^2}{n(n+1)}F_n^m \tag{6.112}$$

as the spectral form of the barotropic vorticity equation (6.105). The first term on the right-hand side of (6.112) is the beta term (the acceleration due to the advection of the earth's vorticity), while the second term is the advection of relative vorticity.

Source: Krishnamurti et al. 2006

The spectral form of the barotropic vorticity equation consists of a set of ordinary nonlinear differential equations in spectral space. It should be noted that F_n^m is a nonlinear term. We use the *transform method* to evaluate such nonlinear terms.

The transform method involves calculating the terms $\partial\psi/\partial\lambda$, $\partial\psi/\partial\mu$, $\partial\zeta/\partial\lambda$, and $\partial\zeta/\partial\mu$ on the grid points by projecting the spectral coefficients onto the space domain. These are then multiplied to get values of the nonlinear terms $F(\lambda,\mu)$ on the grid points. The spectral analysis of $F(\lambda,\mu)$ then involves the Fourier analysis of the $F(\lambda,\mu)$ along latitude circles, followed by a Legendre transform of the resulting Fourier coefficients to obtain the spectrum F_n^m. This procedure is called the *Fourier-Legendre transform*.

We can write

$$F(\lambda,\mu) = \frac{1}{a^2}\left(\frac{\partial\psi}{\partial\mu}\frac{\partial\zeta}{\partial\lambda} - \frac{\partial\psi}{\partial\lambda}\frac{\partial\zeta}{\partial\mu}\right) \tag{6.113}$$

$$= \frac{1}{a^2(1-\mu^2)}\left((1-\mu^2)\frac{\partial\psi}{\partial\mu}\frac{\partial\zeta}{\partial\lambda} - \frac{\partial\psi}{\partial\lambda}(1-\mu^2)\frac{\partial\zeta}{\partial\mu}\right).$$

Let

$$\psi(\lambda,\mu) = \sum_m \sum_n \psi_n^m(t) Y_n^m(\lambda,\mu), \tag{6.114}$$

Source: Krishnamurti et al. 2006

so that upon taking the derivative of (6.114) with respect to μ, multiplying by $(1-\mu^2)$, and remembering that $Y_n^m(\lambda,\mu) = P_n^m(\mu)e^{im\lambda}$ we obtain the north-south derivative of ψ

$$(1-\mu^2)\frac{\partial \psi}{\partial \mu} = \sum_m \sum_n \psi_n^m(t) e^{im\lambda}(1-\mu^2)\frac{\partial}{\partial \mu} P_n^m(\mu), \tag{6.115}$$

where by the recurrence relation (6.38) we have

$$(1-\mu^2)\frac{d}{d\mu} P_n^m(\mu) = -n \in_{n+1}^m P_{n+1}^m(\mu) + (n+1) \in_n^m P_{n-1}^m(\mu)$$

and

$$\in_n^m = \left(\frac{n^2-m^2}{4n^2-1}\right)^{1/2}.$$

Hence using the recurrence relation in equation (6.115) we obtain

$$\begin{aligned}(1-\mu^2)\frac{\partial \psi}{\partial \mu} = \sum_m \sum_n \psi_n^m \big[&-n \in_{n+1}^m P_{n+1}^m(\mu) \\ &+(n+1) \in_n^m P_{n-1}^m(\mu)\big] e^{im\lambda}\end{aligned} \tag{6.116}$$

where the term in brackets is always known.

If we differentiate (6.114) with respect to λ, we obtain the east-west derivative of ψ

Source: Krishnamurti et al. 2006

$$\frac{\partial \psi}{\partial \lambda} = \sum_m \sum_n im\psi_n^m(t)P_n^m(\mu)e^{im\lambda}. \qquad (6.117)$$

Using (6.116) and (6.117) we can obtain the grid-point values of $\partial\psi/\partial\lambda$ and $(1-\mu^2)\partial\psi/\partial\mu$. Similarly, the grid-point values of $(1-\mu^2)\partial\zeta/\partial\mu$ and $\partial\zeta/\partial\lambda$ are calculated from the spectral coefficients of vorticity ζ_n^m. Having obtained $(1-\mu^2)\partial\psi/\partial\mu$, $\partial\psi/\partial\lambda$, $(1-\mu^2)\partial\zeta/\partial\mu$, and $\partial\zeta/\partial\lambda$ in grid space, we obtain $F(\lambda,\mu)$ from (6.113) in grid space. The Fourier-Legendre transform is then applied to $F(\lambda,\mu)$ to obtain F_n^m.

Knowing the spectral coefficients ψ_n^m and F_n^m, one can calculate the term on the right-hand side of the spectral form of vorticity equation (6.112). This gives us $d\psi_n^m/dt$, the time tendency of ψ_n^m, which, along with the value of ψ_n^m at the previous time step, can be used to obtain ψ_n^m at a future time step. In practice, the following steps are needed to integrate the barotropic vorticity equation:

Step 1: From the coefficients ψ_n^m and ζ_n^m, obtain the grid-point values of $(1-\mu^2)\partial\psi/\partial\mu$ and $(1-\mu^2)\partial\zeta/\partial\mu$ along a latitude circle using (6.116).

Step 2: Similarly, using (6.117), obtain grid-point values of $\partial\psi/\partial\lambda$ and $\partial\zeta/\partial\lambda$ along the latitude circle.

Source: Krishnamurti et al. 2006

Step 3: Multiply $(1-\mu^2)\partial\psi/\partial\mu$, $\partial\zeta/\partial\lambda$, $\partial\psi/\partial\lambda$ and $(1-\mu^2)\partial\zeta/\partial\mu$ to calculate

$$F(\lambda,\mu) = \frac{1}{a^2}\left(\frac{\partial\psi}{\partial\mu}\frac{\partial\zeta}{\partial\lambda} - \frac{\partial\psi}{\partial\lambda}\frac{\partial\zeta}{\partial\mu}\right)$$
$$= \frac{1}{a^2(1-\mu^2)}\left((1-\mu^2)\frac{\partial\psi}{\partial\mu}\frac{\partial\zeta}{\partial\lambda} - \frac{\partial\psi}{\partial\lambda}(1-\mu^2)\frac{\partial\zeta}{\partial\mu}\right)$$

on the grid points (λ,μ).

Step 4: Perform the Fourier transform of $F(\lambda,\mu)$ along the latitude circle to obtain the Fourier components $F^m(\mu)$. This is done by using the FFT.

Step 5: Perform the Legendre transform of $F^m(\mu)$ at the various latitude to obtain F_n^m, the spherical harmonic amplitudes of $F(\lambda,\mu)$. This is done by using Gaussian quadrature.

Step 6: From ψ_n^m and F_n^m, calculate the right-hand side of the spectral vorticity equation and obtain $d\psi_n^m/dt$.

Step 7: From $\psi_n^m(t-\Delta t)$ and $d\psi_n^m/dt$, obtain $\psi_n^m(t+\Delta t)$ at the next time step as

$$\psi_n^m(t+\Delta t) = \psi_n^m(t-\Delta t) + 2\Delta t\frac{d\psi_n^m}{dt}.$$

Source: Krishnamurti et al. 2006

Thanks for your kind attention!